①

Department of Mathematics and Statistics, Concordia University

Math 203: Calculus (I)

Midterm Exam, Winter Semester, 2007

Instructor: **Dr. Ming Mei**

NOTE: `Calculators are not allowed.`

---

1. [20pts]

   (a) Suppose $f(x) = \sqrt{1-x}$ and $g(x) = \sin^2 x$. Find $f \circ g \circ f$ and $g \circ f \circ g$. Simplify.

   (b) Find the inverse of the function $f(x) = 2 + x^5$. Determine the domain and the range of f and f^{-1}, respectively.

2. [20pts] Evaluate the limits.

   $$\text{(a)} \quad \lim_{x \to 5} \frac{\sqrt{x-4}-1}{x^2-4x-5} \qquad \text{(b)} \quad \lim_{x \to \infty} \frac{4x(x^2+1)^4}{(3x^3+2)^2(2x^2+1)}.$$

3. [20pts]

   (a) Consider the function $f(x) = \dfrac{|x-2|}{x-2}$. Calculate both one-sided limits at the point where the function is undefined.

   (b) Find parameters a and b such that the function

   $$f(x) = \begin{cases} -\frac{4}{x^2}, & \text{if } x \le -1, \\ ax - b, & \text{if } -1 < x \le 0, \\ x^2 - 2, & \text{if } x > 0 \end{cases}$$

   will be continuous at every point.

4. [40pts] Find derivatives of the following functions (do not simplify the answer):

   (a) $f(x) = \left(x + \dfrac{1}{x}\right)^2 \ln 5x;$

   (b) $f(x) = \dfrac{\sin^2 x}{1+x^2};$

   (c) $f(x) = \cos^2(x - e^{x^2});$

   (d) $f(x) = x^{\arctan x^2}$ (use logarithmic differentiation).

Solutions to the Midterm Test

by. M. Mei

1 (a).

$$f\circ g\circ f = f(g(f(x)))$$
$$= \sqrt{1-g(f(x))}$$
$$= \sqrt{1-\sin^2(f(x))}$$
$$= \sqrt{1-\sin^2\sqrt{1-x}}$$
$$= \sqrt{\cos^2\sqrt{1-x}}$$
$$= \left|\cos\sqrt{1-x}\right|. \quad //$$

$$g\circ f\circ g = \sin^2(f(g(x)))$$
$$= \sin^2\sqrt{1-g(x)}$$
$$= \sin^2\sqrt{1-\sin^2 x}$$
$$= \sin^2\sqrt{\cos^2 x}$$
$$= \sin^2|\cos x|. \quad //$$

1 (b)

$$y = f(x) = 2 + x^5$$
$$\downarrow \qquad\qquad \downarrow$$
$$x \quad = 2 + y^5, \text{ i.e. } y = (x-2)^{\frac{1}{5}}$$

So, the inverse: $f^{-1}(x) = \sqrt[5]{x-2}$

Domain of f: $(-\infty, \infty)$

Range of f: $(-\infty, \infty)$

Domain of f^{-1}: $(-\infty, \infty)$

Range of f^{-1}: $(-\infty, \infty)$

2 (a) $$\lim_{x\to 5} \frac{\sqrt{x-4}-1}{x^2-4x-5} = \lim_{x\to 5} \frac{\sqrt{x-4}-1}{(x-5)(x+1)} \cdot \frac{\sqrt{x-4}+1}{\sqrt{x-4}+1}$$

$$= \lim_{x\to 5} \frac{(\sqrt{x-4})^2 - 1^2}{(x-5)(x+1)(\sqrt{x-4}+1)}$$

$$= \lim_{x\to 5} \frac{\cancel{\sqrt{x-4}}\ \cancel{x-5}}{\cancel{(x-5)}(x+1)(\sqrt{x-4}+1)}$$

$$= \lim_{x\to 5} \frac{1}{(x+1)(\sqrt{x-4}+1)}$$

$$= \frac{1}{(5+1)(\sqrt{5-4}+1)} = \frac{1}{6\cdot 2} = \frac{1}{12}$$ //

2 (b) $$\lim_{x\to\infty} \frac{4x(x^2+1)^4}{(3x^3+2)^2(2x^2+1)}$$

$$= \lim_{x\to\infty} \frac{4x(x^2+1)^4/x^9}{(3x^3+2)^2(\cancel{2}x^2+1)/x^9}$$ (Highest order is 9)

$$= \lim_{x\to\infty} \frac{\frac{4x}{x}\ \frac{(x^2+1)^4}{x^8}}{\frac{(3x^3+2)^2}{x^6}\cdot\frac{(2x^2+1)}{x^2}\cdot\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{4\ \left(\frac{x^2+1}{x^2}\right)^4}{\left(\frac{3x^3+2}{x^3}\right)^2\left(\frac{2x^2+1}{x^2}\right)\cdot\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{4\left(1+\frac{1}{x^2}\right)^4}{\left(3+\frac{2}{x^3}\right)^2\left(2+\frac{1}{x^2}\right)\cdot\frac{1}{x}}$$

$$= \frac{4\,(1+0)^4}{(3+0)^2\,(2+0)\cdot 0} = \frac{4}{0} = \infty$$

"DNE" //

3 (a)

$$\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{\cancel{x-2}}{\cancel{x-2}} = 1$$

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(\cancel{x-2})}{\cancel{x-2}} = -1$$

//

3 (b) Since $f(x)$ is needed to be continuous, we have:

$f(-1^-) = f(-1^+)$, and

$f(0^-) = f(0^+)$.

i.e.

$$\begin{cases} \frac{-4}{(-1)^2} = a(-1) - b \\ a\cdot 0 - b = 0^2 - 2 \end{cases}$$

which gives:

$a = 2$, $b = 2$.

//

4 (a)

$$f'(x) = \left(\left(x+\tfrac{1}{x}\right)^2\right)' \ln 5x + \left(x+\tfrac{1}{x}\right)^2 (\ln 5x)'$$

(product rule)

$$= 2\left(x+\tfrac{1}{x}\right)^{2-1} \cdot \left(x+\tfrac{1}{x}\right)' \ln 5x$$

$$+ \left(x+\tfrac{1}{x}\right)^2 \frac{1}{5x} \cdot (5x)'$$

(chain rule)

$$= 2\left(x+\tfrac{1}{x}\right)\left(1+(-1)x^{-1-1}\right) \ln 5x$$

$$+ \left(x+\tfrac{1}{x}\right)^2 \cdot \frac{1}{5x} \cdot 5$$

$$= 2\left(x+\tfrac{1}{x}\right)\left(1-\tfrac{1}{x^2}\right) \ln 5x$$

$$+ \left(x+\tfrac{1}{x}\right)^2 \cdot \frac{1}{x}$$

//

4 (b)

$$f'(x) = \left(\frac{\sin^2 x}{1+x^2}\right)' = \frac{(\sin^2 x)'(1+x^2) - \sin^2 x\,(1+x^2)'}{(1+x^2)^2}$$

$$= \frac{2\sin x\,(\sin x)'(1+x^2) - \sin^2 x\,(0+2x)}{(1+x^2)^2}$$

$$= \frac{2\sin x\cos x\,(1+x^2) - 2x\sin^2 x}{(1+x^2)^2}$$

//

4 (c). $f'(x) = 2\left[\cos(x-e^{x^2})\right]^{2-1} \cdot \left(\cos(x-e^{x^2})\right)'$

(Chain Rule)

$= 2\cos(x-e^{x^2}) \cdot \left(-\sin(x-e^{x^2})\right) \cdot (x-e^{x^2})'$

(Chain rule)

$= -2\cos(x-e^{x^2})\sin(x-e^{x^2}) \cdot \left(1-e^{x^2} \cdot (x^2)'\right)$

(Chain rule)

$= -2\cos(x-e^{x^2})\sin(x-e^{x^2}) \cdot (1-2xe^{x^2})$

$= -\left[\sin\left(2(x-e^{x^2})\right)\right] \cdot (1-2xe^{x^2})$ //

4 (d)

$$y = x^{\arctan x^2}$$

Taking logarithms to the above equation:

$$\ln y = \ln x^{\arctan x^2}$$

$$= \arctan x^2 \cdot \ln x$$

Taking differentiation to the both sides of the above equation, we have:

$$\frac{1}{y} y' = (\arctan x^2)' \ln x + \arctan x^2 (\ln x)'$$

chain rule

$$= \frac{1}{1+(x^2)^2} \cdot (x^2)' \ln x$$

$$+ \arctan x^2 \cdot \frac{1}{x}$$

$$= \frac{2x}{1+x^4} \ln x + \frac{1}{x} \arctan x^2$$

So, $y' = y \left[\frac{2x}{1+x^4} \ln x + \frac{1}{x} \arctan x^2 \right]$

$$= x^{\arctan x^2} \left[\frac{2x \ln x}{1+x^2} + \frac{\arctan x^2}{x} \right]$$

//